

Lesson 7

Calculating Confidence Intervals for Proportions

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In Lesson 6 we learned the basic ideas for one type of inferential statistics, using a proportion obtained from a sample to estimate the corresponding proportion for the entire population. As a reminder, our primary example was this, as reported in the news media: “According to a recent survey, 54% of Americans age 18 and above believe that the amount of taxes they pay is fair, with a margin of error $\pm 4.2\%$.” Here are a few of the crucial ideas from that lesson (and earlier lessons):

- Researchers wanted to make a statement about the *population* consisting of *all* Americans age 18 and above.
- To do this, they chose a *sample* of people from that population.
- They measured the proportion, \hat{p} , for this sample. This proportion is a *statistic*, because it is a fact about the sample. The value of \hat{p} was 54%.
- They used this statistic to make a statement about p , the corresponding proportion for the entire population. Because p is a fact about the population, it is called a *parameter*.
- The statement consisted of two parts:
 - A *point estimate* – the estimate that p is 54%.
 - An *interval estimate*, created by combining the point estimate with a margin of error. In this case, they estimate that p is 54%, with a margin of error $\pm 4.2\%$. In other words, they estimate that p is somewhere in the interval $(54\% - 4.2\%, 54\% + 4.2\%)$, which is equal to $(49.8\%, 58.2\%)$.
- They used a method which gets the right answer 95% of the time for the size of sample they used, so this interval is called a *95% confidence interval*. The *confidence level* for this interval estimate is 95%.
- The mathematics of what they did is valid, but it depends on obtaining a *simple random sample*. To the extent that the sample was not truly random, the results become less reliable. That is, the results may turn out to be incorrect more than 5% of the time if the sample was not randomly chosen.

In the last section of Lesson 6, we explained very briefly why a margin of error of about $\pm 3\%$ gives correct answers about 95% of the time for samples of about size 1000. The purpose of the current lesson is to learn how to be more precise in our calculation of the appropriate margin of error. In addition, we want to be able to determine the margin of error for different sample sizes (other than $n = 1000$) and for different confidence levels (other than 95%). The methodology is based on the sampling distribution of the sample proportion, just as described in the previous lesson.

7.1 – A Quick Review: the Sampling Distribution

Creating a confidence interval with a particular confidence level is based on knowing what happens when a random sample is taken from a population with a particular population proportion p . Mathematicians have answered that question using mathematical calculations well beyond the scope of these lessons. As we described in Lesson 6:

Reminder: The theoretical histogram containing the \hat{p} values for *all possible* samples of the chosen size is called the **sampling distribution of the sample proportions**. If it is clear that we are talking about sample proportions, we will simply refer to the **sampling distribution**.

Mathematicians have established three fundamental facts about this theoretical sampling distribution (the sampling distribution of the sample proportions):

- The mean of the distribution is equal to the population proportion – what we have labeled with the variable p .
- The standard deviation of the distribution can be calculated using the formula $\sqrt{\frac{p(1-p)}{n}}$
- The distribution is mound shaped. In fact, provided n is large enough, the theoretical sampling distribution is not just mound-shaped, it is in fact approximately normal.

To help us visualize what the mathematicians tell us, in Lesson 6 we used the applet at the following link to experiment with the process of repeatedly taking samples from a population with proportion p , recording the sample proportion \hat{p} for each sample, and making a histogram of the resulting \hat{p} values.

[Histogram of p-hat values](#)

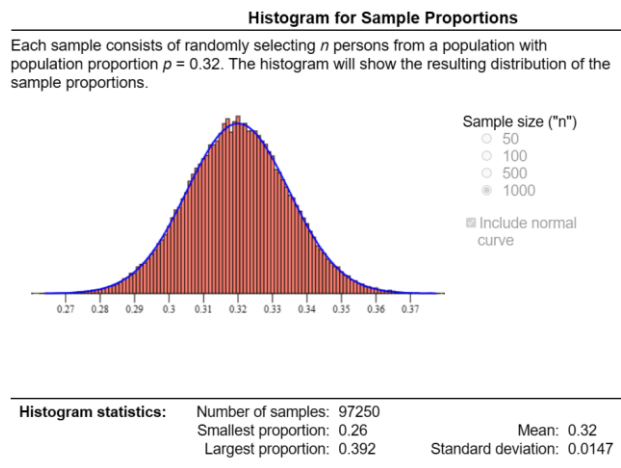
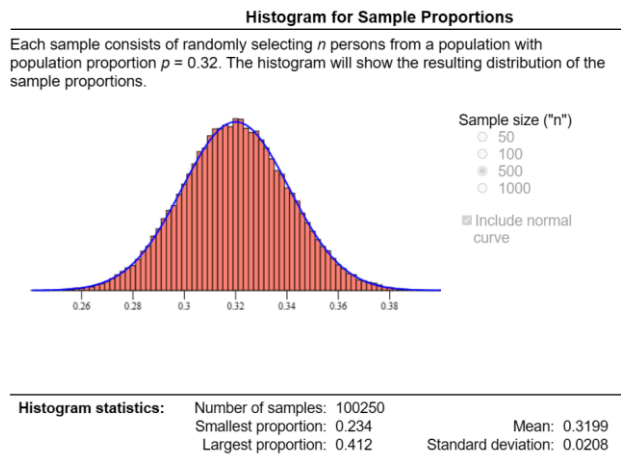
In the applet you can select a sample size n , with the default being 1000. When you click on the “Do 250 samples” button the applet will take 250 samples of size n and create the corresponding histogram. You can use the controls to add more samples to the graph, or to start the whole process over. If you wish, you can use the check box to overlay a normal curve on the resulting histogram. You can also change the sample size to 50, 100, or 500 while starting over¹,

Example. Here are the results obtained by the author using this applet. The first graph used samples of size $n = 500$, the second $n = 1000$. In each case the population proportion was 0.32 or 32%, and in

¹ NOTE: We actually have two versions of the applet. In the first, the scale for the histograms is based on the range of values in the histogram, so the graphs for $n = 50$, 100, and so on, look quite similar. In the second, at the following link, all the graphs are drawn on the same scale. Using the second applet will emphasize how much closer the various sample proportions are to the mean when the sample size is larger. Here is the link:

[Histogram of p-hat values](#)

each case the author generated approximately 100,000 samples, overlaying the resulting histogram with a normal curve.



For each run of the applet, observe the following:

- The histogram matches the overlaid normal curve quite closely.
- The mean for the histogram is very close to the population mean of 0.32; in the first run we got 0.3199, in the second 0.32.
- Rounded to four places, the formula the mathematicians have developed for the standard deviation (standard error) of the sampling distribution, $se = \sqrt{\frac{p(1-p)}{n}}$, yields, and $se = \sqrt{\frac{0.32(1-0.32)}{500}} = 0.0209$ for the first run and $se = \sqrt{\frac{0.32(1-0.32)}{1000}} = 0.0148$ for the second run. The results we obtained in our simulation runs (0.0208 and 0.0147) match this very closely.

Exercise 1². Use the applets to experiment. After opening the applet, choose a sample size and use the start sampling / stop sampling buttons to generate about 100,000 samples with an overlaid normal curve. Each time you run the applet, record your answers to these questions:

- What is the population proportion p for this run?
- Calculate the standard error for the sampling distribution using the formula $s.e. = \sqrt{\frac{p(1-p)}{n}}$
- Does the histogram match the normal curve fairly closely?
- What is the mean for the histogram? Is it close to p ?
- What is the standard deviation for the histogram? Is it close to what you calculated in step b?

² Solutions to the exercises may be found at the end of the lesson.

7.1 – 95% Confidence Intervals

In the last section of Lesson 6, we explained very briefly why a margin of error of about $\pm 3\%$ gives correct answers about 95% of the time for samples of about size 1000. To summarize, the reason is this:

Reminder: When we are working with a sampling distribution, it is common to use the term *standard error* (abbreviated *se* or sometimes *SE*) as a synonym for standard deviation.

Suppose you create a histogram showing the value of \hat{p} for all the possible samples of size n from the population (called the “sampling distribution of the sample proportion”). This histogram will be mound shaped with a mean equal to the population proportion p and a standard deviation given by $se = \sqrt{\frac{p(1-p)}{n}}$.

Because of the mound shape, about 95% of the samples will be within 2 standard deviations of the mean. This implies that for 95% of the samples the distance between the sample proportion \hat{p} and the population proportion p will be less than $2\sqrt{\frac{p(1-p)}{n}}$.

So, if you use a margin of error of $2 \cdot se = 2\sqrt{\frac{p(1-p)}{n}}$, you will be correct about 95% of the time.

When n is 1000, this formula always gives a value pretty close to 3%. So $\pm 3\%$ is a reasonably good margin of error to use when n is 1000.

In this section we will learn how to be more precise in our calculation of the appropriate margin of error. In addition, we want to be able to determine the margin of error for different sample sizes (other than $n = 1000$). In the next section we will see what is necessary to calculate confidence intervals for different confidence levels (other than 95%).

Provided the sample is large enough³, the sampling distribution of the sample proportions is not just mound shaped but indeed approximately normal. This means that we can use properties of a normal distribution to calculate a good value for the margin of error. To get started, let’s do a very quick review of some ideas first discussed in Lesson 4.

³ How large is “large enough”? The general rule is that your sample must contain at least 15 *successes* (“yes” answers to the question) and 15 *failures* (“no” answers). In terms of the variables n and \hat{p} , we can write two conditions:

$$n\hat{p} \geq 15$$

that is, #successes is at least 15

$$n(1 - \hat{p}) \geq 15$$

that is, #failures is at least 15

Recall that a z -score is simply a measure of the distance from a data item to the mean, measured in terms of standard deviations. In terms of z -scores, the empirical rule states that for any mound-shaped distribution:

- Approximately 95% of the data has a z -score between -2 and $+2$.

This means that approximately 95% of the data lies between the mean minus two standard deviations and the mean plus two standard deviations. This was the basis for our statement that $2 \cdot se = 2\sqrt{\frac{p(1-p)}{n}}$ is a reasonable value for the margin of error if you want to be correct 95% of the time.

In Lesson 4 we learned that we can be more precise when we have a normal distribution. We can use [Table A](#) to find a z -score with the property that 2.5% of the area lies to the left of that z -score. Here is the pertinent part of Table A:

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294

We see that the z -score -1.96 has 2.5% of the area to the left of that z -score. By symmetry, 2.5% of the area is to the right of $z = 1.96$. Therefore, for a normal distribution we can state that:

- 95% of the data has a z -score between -1.96 and $+1.96$

Based on this fact, we will calculate the margin of error as 1.96 standard deviations, that is $1.96 \cdot se = 1.96\sqrt{\frac{p(1-p)}{n}}$. If we do this, our interval estimate will be correct 95% of the time. That is, our interval will be a 95% confidence interval.

There is only one problem with this formula. How can we calculate a margin of error using a formula that contains p , the proportion for the entire population? We *don't know* the proportion for the entire population. The whole point of taking a sample is to try to obtain an estimate for the unknown population proportion.

What shall we do? Fortunately, if we just do the best we can, the results will still be good. We don't know p , the population proportion, but we do have an estimate for it. Let's just use that estimate, \hat{p} , the proportion from the sample. So we will use $1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ as the margin of error. The good news, as illustrated by the following exercise, is that this doesn't change the result by very much at all – so we will still be correct about 95% of the time.

- Exercise 2:** Suppose that for a sample of size 850, your sample proportion is $\hat{p} = 54\%$, but the population proportion is actually $p = 56.7\%$.
- Calculate the margin of error you *should* use for a 95% confidence interval.
 - Do the same calculation but using the sample proportion instead. Is the answer much different?

An aside on terminology. The standard error (that is, standard deviation) for the sampling distribution is given by the formula $se = \sqrt{\frac{p(1-p)}{n}}$. However, when we are calculating confidence intervals, we do not know the value for p (the proportion in the entire population). We do the best we can – we use \hat{p} , the proportion in the sample. Our formula becomes $se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. The first formula gives the “exact standard error,” the second the “approximate standard error.” Textbooks differ in how they refer to these quantities. For our purposes, we will just use the generic term “standard error,” realizing that if we do not know p we must instead use \hat{p} in the calculation. Using this notation, the margin of error, m.e., is given by the formula

$$\text{m.e.} = 1.96 \cdot se = 1.96 \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Calculating a 95% confidence interval

Here is a summary of the process we use to calculate a 95% confidence interval. Our goal is to estimate a population proportion p , by taking a random sample from the population. Let n indicate the size of the sample, and let \hat{p} be the proportion for the sample.

- The point estimate is \hat{p} .
- We calculate the (approximate) standard error using $se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.
- We calculate the margin of error using $m.e. = 1.96 \cdot se$
- We can write the interval as $(\hat{p} - m.e., \hat{p} + m.e.)$ which is the same as $(\hat{p} - 1.96 se, \hat{p} + 1.96 se)$.
- This is a 95% confidence interval. That is, the method gives a correct answer for 95% of the possible samples.
- To use this method, n must be “large enough”:

$n\hat{p} \geq 15$	that is, #successes is at least 15
$n(1 - \hat{p}) \geq 15$	that is, #failures is at least 15

Example: Give a point estimate and a 95% confidence interval for a population proportion, based on a sample in which 753 of the 1077 people in the sample respond yes to the question posed by the researchers.

Solution: The point estimate is the proportion we obtain from the sample, that is, the sample proportion:

$$\hat{p} = \frac{\text{count for sample}}{\text{sample size}} = \frac{753}{1077} = 0.6992$$

Our best estimate is that, if we asked the same question to the entire population, 69.92% of them would answer yes.

Because this is only an estimate, we also want to calculate the margin of error. This is a two step process. First we calculate the standard error:

$$se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.6992(1-0.6992)}{1077}} = 0.0140$$

Then we use the standard error to calculate the margin of error:

$$m.e. = 1.96 \cdot se = 1.96(0.0140) = 0.0274$$

We could report the results as 69.92%, with a margin of error of 2.74%. Or we could calculate the confidence interval by subtracting and adding the margin of error to the sample proportion:

$$(0.6992 - 0.0274, 0.6992 + 0.0274) = (0.6718, 0.7266)$$

Writing this as percents, we would write (67.18%, 72.66%). We are 95% confident that between 67.18% and 72.66% of the entire population would answer yes to the question.

Note: To justify the use of the normal distribution, both the number of “successes” and the number of “failures” must be at least 15. The number of successes is the number of yes answers, namely 753. The number of failures is a count of the others in the sample, $1077 - 753 = 324$. Both are certainly much larger than 15, so the method is justified.

Exercise 2: Estimate the population proportion (using a point estimate and a 95% confidence interval) for each of the following.

- In a random sample from the population, 382 of the 779 people answer yes to the question posed.
- The researchers obtain 519 yes answers and 982 no answers from the sample.

The applet at the following link gives additional practice calculating point estimates and 95% confidence intervals.

[95% confidence intervals](#)

7.2 – Interpretation for Confidence Intervals

Stating the results, in the context of the problem

There is another important issue to be addressed: *How do we report the results?* We have already seen one way, using the margin of error terminology:

According to a recent survey, 54% of Americans age 18 and above believe that the amount of taxes they pay is fair, with a margin of error $\pm 4.2\%$.

This is the method usually used in the media. One problem with this way of reporting the result is that it does not mention the confidence level. Here is a reporting method that does contain the confidence level:

We are 95% confident that between 49.8% and 58.2% of all Americans age 18 and above believe that the amount of taxes they pay is fair.

This method gives both the interval and the confidence level. We have included the word “all” to emphasize that this is a statement about the entire population, not just about the people in the sample. In your own work, it is a good idea to include the word “all” to emphasize exactly what the population is. Here is a general template that can be used:

We are 95% confident that between _____% and _____% of _____ describe the population _____ describe what a “yes” answer to the question indicates

Example. A recent report by the Centers for Disease Control states that 16.8% of American adults are smokers. The author was recently taking care of family business in the town where his in-laws live, and felt that he was seeing more people smoking than he usually did in his own home town. This raised the question: Is the proportion of smokers in his in-laws’ town 16.8%, or is it higher (or possibly lower)?

To answer this question, one could survey a random sample from that town. Let us suppose that we did so, and that 215 out of 1100 people were smokers⁴. We can calculate a 95% confidence interval using these steps:

$$\hat{p} = \frac{215}{1100} = 0.1955$$

$$se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.1955(1-0.1955)}{1100}} = 0.0120$$

$$m.e. = 1.96 \cdot se = 1.96(0.0120) = 0.0235$$

$$(0.1955 - 0.0235, 0.1955 + 0.0235) = (0.1720, 0.2190) = (17.2\%, 21.9\%)$$

These results could be reported using “margin of error” terminology as:

According to a recent survey, the proportion of smokers in that town is 19.55%, with a margin of error of $\pm 2.35\%$.

To include the confidence level and write the result as an interval we might report the results as

We are 95% confident that between 17.2% and 21.9% of all the people in that town are smokers.

⁴ The examples in these notes generally fall into one of several categories. Some may be taken completely from actual studies reported in the literature. Others are adaptations of actual studies, with the data modified to allow for practice with the methods being studied. Still others are situations which are similar to studies that statisticians do but which are made up by the author. This example falls into the latter category (although the 16.8% figure reported by the CDC is real).

Interpreting reported population proportions

When an estimate of a population proportion is reported, there are additional things to keep in mind. For example, consider our primary example: “54% of Americans age 18 and above believe that the amount of taxes they pay is fair, with a margin of error $\pm 4.2\%$.” This simple sentence conveys a lot of unstated information.

- 54% of the people surveyed answered yes.
- Although I didn’t ask everyone in the entire country, I’m using the result from my survey to *estimate* that for the entire country 54% would also answer yes.
- I *think* the actual value is close to this estimate – no further away than $\pm 4.2\%$. Put another way, I *think* the actual value is somewhere between 49.8% and 58.2%.

The news media do not always give the margin of error when they report a population proportion. They might simply state something similar to this: A recent survey showed that 43% of all Republicans favor Candidate X. When you hear such a report, you should always keep in mind that there is a margin of error implicit in the results.

There is even more to the story, although the news media almost never reports this part. There is the confidence level. For our example, if you go back to the original report you find a confidence level of 95% reported, although that didn’t make it into the news media’s version. So there is also this:

- When I say I *think* the actual value is somewhere between 49.8% and 58.2%, I am 95% confident that my result is correct. This is because I used a method that gives a correct answer 95% of the time.
- Since the method gives a correct answer 95% of the time, this means that 5% of the time the method yields an incorrect answer. So 5% of the time the method yields an estimate that is *not* within $\pm 4.2\%$ of the actual population proportion.

To the extent that the sampling method was indeed random, and provided we correctly interpret the results, the methodology is sound. But we should be careful not to hear only the “54%” in the report, but also the rest of the story.

“Plausible” values

In the example, when we indicate the confidence interval as (49.8%, 58.2%), we can think of the numbers in the interval as the “most plausible” or “most believable” possible values for the population proportion. That is not to say that the actual proportion *cannot* be outside the interval – after all, this could be one of those intervals which gets an incorrect answer. However, since this is a 95% confidence interval, only 5% of all the possible intervals would give an incorrect answer.

As a result, if we ask ourselves the question, “Based on this interval, is it plausible (believable, feasible) that the population proportion could be 49%?” our answer will be, “No.” *Based on the interval*, we believe the value is between 49.8% and 58.2%, and 49% is not between those two numbers.

On the other hand, if we ask, “Based on this interval, is it plausible that the population proportion could be 58%?” the answer will be, “Yes” because 58% is one of the values within the interval.

Example. Researchers select a random sample from a certain population, and ask a yes/no question. In the sample, 513 answer yes and 627 answer no. They report a point estimate and a 95% confidence interval for the proportion of the population that would answer yes. The point estimate is 0.4500 and the confidence interval is (0.4211, 0.4789). Based on these results,

- Is it plausible that in the entire population 47% of the people would answer yes to the question?
- Is it plausible that in the entire population 42% of the people would answer yes to the question?
- Can we conclude that less than half of the population would answer yes to the question?
- Would it be correct to state that the population proportion is 0.4500?
- There is a chance that the researchers have reached an incorrect conclusion, but that chance is not very large.

Solution

- Because 47% (that is, 0.4700) is one of the numbers in the interval, it is plausible that 47% of the entire population would answer yes.
- No. Because 42% (that is, 0.4200) is **not** within the interval, it is not a plausible value, based on these results. **Note:** The question of plausibility is based solely on whether the proportion in question is within the interval.
- Yes – all the plausible proportions in the interval are less than one half.
- No. It is correct to state that the proportion in the **sample** is 0.4500, but that does not imply that the proportion in the entire **population** is 0.4500. It would, however, be correct to state that the population proportion *could be* 0.4500, since 0.4500 is one of the plausible values in the interval – but it also *could be* 0.4253, or 0.4788, or in fact any number found within the interval.
- True. The methodology used to calculate the results yields an interval which does in fact contain the true population proportion 95% of the time (that is the meaning of “95% confidence”). So the results could be incorrect – 5% of the time the methods yield incorrect results. (It is worth noting that generally speaking, even the intervals which “miss” typically do not miss by very much, however.)

Example. A researcher wants to know if college students are in favor of a textbook rental policy. She asks the question to 843 randomly chosen college students and 76% say “yes”. Which of the following statements are correct based on this description.

- The population consists of the 843 randomly chosen college students.
- Using this sample, we can be very confident that more than half of all college students would say “yes” to this question.
- The 76% corresponds to the variable \hat{p} .
- The population consists of all college students who favor a textbook rental policy.
- The population consists of all college students.
- If another researcher did a similar study with a sample of 843 college students, they would get the same percentage (76%) because the sample is the same size.

Solution

- Incorrect. It would be correct to state that the **sample** consists of the 843 randomly chosen college students.
- Correct. Without doing the calculations, we realize that if we calculate a confidence interval all the numbers in the interval will be quite a bit larger than 50%.
- Correct. The variable \hat{p} refers to the proportion in the sample.
- Incorrect. See part (e) below.

- e. Correct. We have two ways to come to this conclusion. The first sentence of the description indicates the researcher wonders if *college students* favor the policy, the second sentence indicates that the sample consists of randomly chosen *college students*. In general, if a sample consists of n individuals chosen from a certain group, the population is *all* individuals in that same group.
- f. Incorrect. If another researcher did a similar study with a sample of 843 college students, we would expect to see similar results (that is, a value reasonably close to 76%), but not necessarily exactly the same percentage.

The following applet provides additional practice in calculating confidence intervals, along with interpreting the results.

[Calculations with questions on plausible values](#)

For additional practice in interpreting confidence intervals, you may use these two applets.

[Questions on plausible values](#)

[Other interpretation questions](#)

7.3 – Other Confidence Levels

It is quite simple to obtain confidence intervals for other confidence levels, if we keep in mind the basis for the 95% confidence interval. In general, the confidence interval will have this form:

$(\hat{p} - m.e., \hat{p} + m.e.)$, where:

- $m.e. = z^* \cdot se$ (for 95% confidence intervals this formula is $m.e. = 1.96 \cdot se$)
- $se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
- z^* is chosen based on the desired confidence level

That is, we form the interval by using the sample proportion as the point estimate, then adding and subtracting a margin of error, $z^* \cdot se$, where z^* is chosen to achieve the desired confidence level. For a normal distribution, 95% of the data lies within 1.96 standard deviations of the mean, so we have been using $z^* = 1.96$.

Now suppose we want a 98% confidence interval. We simply ask this question for a normal distribution: “For what z^* does 98% of the data lie within z^* standard deviations of the mean?” Put another way, we must fill in the blanks in this statement:

- 98% of the data in a normal distribution has z -score between _____ and _____.

We have worked this problem as an exercise back in Lesson 4. There will be 1% of the data to the left of $-z^*$ and 1% to the right of $+z^*$. We look for 1% = 0.0100 in Table A. We do not find this exact value, but the closest we do find is 0.0099, for $z = -2.33$.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110

So we choose $z^* = 2.33$.

Exercise 3: Give a 94% confidence interval based on a sample for which the sample proportion is 70% and the sample size is 280.

We can use Table A, as illustrated, to determine the appropriate value for z^* . However, for the most common confidence levels there is a shortcut, provided by [Table B](#). This table lists values for z^* to three decimal places, in the bottom row of the table. Here is the pertinent part of Table B:

	Confidence Level							
	80%	90%	95%	96%	98%	99%	99.5%	99.9%
	Right-tail Probability							
df	0.10	0.05	0.025	0.02	0.01	0.005	0.0025	0.0005
1	3.078	6.314	12.706	15.895	31.821	63.657	127.321	636.619
2	1.886	2.920	4.303	4.849	6.965	9.925	14.089	31.599
3	1.638	2.353	3.182	3.482	4.541	5.841	7.453	12.924
				⋮				
60	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.460
80	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.416
100	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.390
z^*	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.291

Looking at the “Confidence Level” header and the bottom row of the table, we see that z^* should be 1.960 for a 95% confidence interval. For a 98% confidence interval $z^* = 2.326$ (compare this to the rounded value of 2.33 we obtained using Table A).

Example. In a recent survey of 1022 U.S. adults, 500 indicated that they approve of the way the Supreme Court is handling its job. The corresponding 95% confidence interval is (45.86%, 51.99%).

- Based on the 95% confidence interval, which if any of these conclusions are justified?
 - Less than half the adults in the country approve of the way the Supreme Court is handling its job.
 - More than half the adults in the country approve of the way the Supreme Court is handling its job.
- Without doing any calculations, would a 99% confidence interval be larger or smaller than this 95% confidence interval?
- Calculate the 99% confidence interval.
- Interpret the interval “in the context of the problem.”
- Calculate a 90% confidence interval.

Solution.

- Since the interval contains plausible values both below and above 50%, neither of the conclusions is justified. The proportion *could be* less than 50%, but it also *could be* greater than 50%.
- The 99% confidence interval will be larger. There are at least two ways to realize this. First, in order to be more confident in our results, our results will need to include a larger range of plausible values. The smaller the interval, the less confident we become that the true proportion is within the interval. Second, the 99% interval will use a larger z^* (2.576) than the z^* for the 95% interval (1.96).
- $$\hat{p} = \frac{500}{1022} = 0.4892$$

$$se = \sqrt{\frac{0.4892(1 - 0.4892)}{1022}} = 0.0156$$

$$m.e. = 2.576 * 0.0156 = 0.0402$$
 Interval is $(0.4892 - 0.0402, 0.4892 + 0.0402) = (0.4490, 0.5294)$
- We are 99% confident that for the population consisting of all U.S. adults, between 44.90% and 52.94% approve of the way the Supreme Court is handling its job.
- The calculations are the same except that we use $z^* = 1.645$.

$$m.e. = 1.645 * 0.0156 = 0.0257$$
 Interval is $(0.4892 - 0.0257, 0.4892 + 0.0257) = (0.4635, 0.5149)$

Exercise 4: In a recent survey of 1093 people in Pennsylvania, 765 indicated that they favor a tax on companies drilling for natural gas in the state. Give a 99% confidence interval, and interpret the result “in the context of the problem.”

The following applet provides additional practice in calculating confidence intervals with varying confidence levels.

[Confidence interval calculations](#)

For additional practice in interpreting confidence intervals, you may use these two applets.

[Questions on plausible values](#)

[Other interpretation questions](#)

7.4 – Requirements, and Warnings

In Lesson 6 we discussed the issue of whether a researcher is justified in making a statement about an entire population, based on a survey of only a sample from that population. In this section, we remind you of that discussion, along with some additional requirements developed in the current lesson.

- Mathematically, the method is based on using a normal distribution. To do so, the sample size n must be “large enough.” In practice, n is large enough provided:
 - $n\hat{p} \geq 15$ that is, the number of *successes* is at least 15
 - $n(1 - \hat{p}) \geq 15$ that is, the number of *failures* is at least 15

- When the results are reported, the margin of error must be included (either using the phrase “margin of error” or by giving an interval). The researcher (and the media) should not imply that the point estimate is guaranteed to be precisely accurate.

The sampling process actually yields an *interval* within which we believe the population proportion lies. This is true whether or not the margin of error is included in the report.

- The researcher should also report the confidence level. (The media typically does not, presumably because either they do not understand it or they believe their audience will not understand it.)

When you see results reported by the media, you should always keep in mind that the method yields correct answers only ____% of the time, filling in the blank with the confidence level. (We believe that 95% is a common-used confidence level when the level is not reported, but if the confidence level is not reported it is impossible to be certain what it is.)

- All the previous discussion is about the mathematical process, based on the assumption that the sample was a *simple random sample*. The remaining warnings relate to potential problems in the sampling process itself. Refer back to Lesson 5 for more information; following are just a few reminders of possible problems.

- In surveying people, it can be very difficult to obtain true randomness. For example, some people refuse to answer surveys, and some people don’t have phones – just two of the many obstacles to obtaining a truly random sample.

- A related problem occurs when a sample is taken from one group, but the results are reported for another group. (Imagine a presidential election poll which samples from only one state but reports the results as valid for the entire country! This would obviously be problematic.)

- Results can be skewed by the wording of the question. This can be unintentional or, in the case of unethical pollsters, intentional.

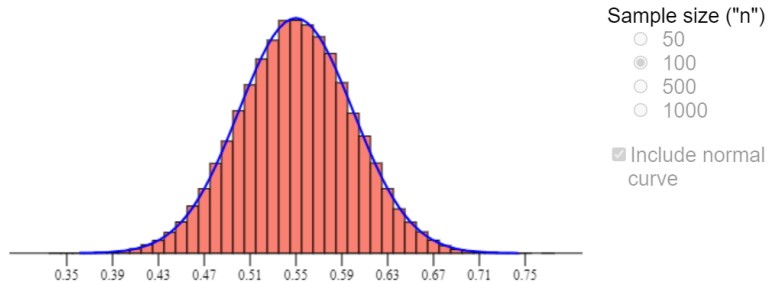
Solutions to Exercises

1: Use the applets to experiment. After opening the applet, choose a sample size and use the start sampling / stop sampling buttons to generate about 100,000 samples with an overlaid normal curve.

Here is one of the author's runs. Your results should be similar.

Histogram for Sample Proportions

Each sample consists of randomly selecting n persons from a population with population proportion $p = 0.55$. The histogram will show the resulting distribution of the sample proportions.



Histogram statistics:	Number of samples: 105250	Mean: 0.55
	Smallest proportion: 0.34	Standard deviation: 0.0498
	Largest proportion: 0.77	

Each time you run the applet, record your answers to these questions:

- a. What is the population proportion p for this run? **0.55**
- b. Calculate the standard error for the sampling distribution using the formula $s.e. = \sqrt{\frac{p(1-p)}{n}}$ **0.0497493719**
- c. Does the histogram match the normal curve fairly closely? **yes**
- d. What is the mean for the histogram? Is it close to p ? **0.55. yes**
- e. What is the standard deviation for the histogram? Is it close to what you calculated in step b? **0.0498. yes**

2: Suppose that for a sample of size 850, your sample proportion is $\hat{p} = 54\%$, but the population proportion is actually $p = 56.7\%$.

- a. Calculate the margin of error you should use for a 95% confidence interval.

$$1.96 \sqrt{\frac{.567(1-.567)}{850}} = 0.0333$$

- b. Do the same calculation but using the sample proportion instead. Is the answer much different?

$$1.96 \sqrt{\frac{.54(1-.54)}{850}} = 0.0335. \text{ The difference is very small, only } 0.0002.$$

- 3: Estimate the population proportion (using a point estimate and a 95% confidence interval) for each of the following.

- a. In a random sample from the population, 382 of the 779 people answer yes to the question posed.

$$\text{The point estimate is } \hat{p} = \frac{382}{779} = 0.4904$$

Here are the calculations for the confidence interval:

$$se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.4904(1-0.4904)}{779}} = 0.0179$$

$$m.e. = 1.96 \cdot se = 1.96(0.0179) = 0.0351$$

$$(0.4904 - 0.0351, 0.4904 + 0.0351) = (0.4553, 0.5255) = (45.53\%, 52.55\%)$$

- b. The researchers obtain 519 yes answers and 982 no answers from the sample.

The sample includes both the yes and the no answers, so $n = 519 + 982 = 1501$. Once we find n , the calculations are the same as for part (a).

$$\hat{p} = \frac{519}{1501} = 0.3458$$

$$se = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.3458(1-0.3458)}{1501}} = 0.0123$$

$$m.e. = 1.96 \cdot se = 1.96(0.0123) = 0.0241$$

$$(0.3458 - 0.0241, 0.3458 + 0.0241) = (0.3217, 0.3699) = (31.17\%, 36.99\%)$$

- 4: Give a 94% confidence interval based on a sample for which the sample proportion is 70% and the sample size is 280.

$$se = \sqrt{\frac{.7(1-.7)}{280}} = 0.0274$$

$$z^* = 1.88 \text{ (look for 0.0300 in Table A, the closest is 0.0301 for } z = -1.88)$$

$$(.7 - 1.88 * 0.0274, .7 + 1.88 * 0.0274) = (.6485, .7515)$$

We are 94% confident that the population proportion lies between 64.85% and 75.15%.

- 5: In a recent survey of 1093 people in Pennsylvania, 765 indicated that they favor a tax on companies drilling for natural gas in the state. Give a 99% confidence interval, and interpret the result “in the context of the problem.”

$$\hat{p} = \frac{765}{1093} = 0.6999$$

$$se = \sqrt{\frac{0.6999(1-0.6999)}{1093}} = 0.0139$$

$$z^* = 2.576 \text{ from Table B}$$

$$(.6999 - 2.576 * 0.0139, .6999 + 2.576 * 0.0139) = (0.6641, 0.7357)$$

We are 99% confident that between 66.41% and 73.57% of the entire state favors a tax on companies drilling for natural gas in Pennsylvania. It is very clear that far more than half the population of the state is in favor of such a tax.